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Extending π-systems to bases of root systems

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Abstract

Let R be an indecomposable root system. It is well known that any root is part of a basis B of R. But when can you extend a set, C, of two or more roots to a basis B of R? A π-system is a linearly independent set of roots such that if α and β are in C, then $\alpha - \beta$ is not a root. We will use results of Dynkin and Bourbaki to show that with two exceptions, $A_3 \subset B_n$ and $A_7 \subset E_8$, an indecomposable π-system whose Dynkin diagram is a subdiagram of the Dynkin diagrams of R can always be extended to a basis of R.

1. Introduction

Let R be an indecomposable root system in a Euclidean space V. A subset B of R is called a basis of R if B is a vector space basis of V and each root of R can be written as a linear combination of roots in B with integral coefficients that are all nonnegative or all nonpositive. It is well known that any root is part of a basis B of R. But when can you extend a set, C, of two or more roots to a basis B of R? A π-system [3,4] is a linearly independent set of roots such that if α and β are in C, then $\alpha - \beta$ is not a root. (It is not assumed to be linearly independent in [4].) A subset of a basis will be

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a π-system, and a π-system will be a basis of a subsystem. We can associate a Dynkin diagram to a π-system, and in order to extend to a basis of R, the Dynkin diagram of the π-system must be a subdiagram of the Dynkin diagram of R. By a subdiagram we mean a diagram obtained by deleting some nodes and their corresponding links. We will assume that the nodes corresponding to short roots are marked, and that the subdiagram preserves the marking. Hence two orthogonal short roots do not form a subdiagram of B_n, while two orthogonal long roots do. We will use results of Dynkin [3] and Bourbaki [2] to show that with two exceptions, $A_3 \subset B_n$ and $A_7 \subset E_8$, an indecomposable π-system whose Dynkin diagram is a subdiagram of the Dynkin diagrams of R can always be extended to a basis of R. Our techniques can easily handle the decomposable case, too, but the results become more tedious to state, and we feel that it would distract from the main ideas of the paper.

2. Results from Dynkin and Bourbaki

If C is a set of roots, then $[C]$ denotes the set of all roots in R that are linear combinations of the roots in C. Let Π be a π-system, and let Π' be a π-system obtained by adjoining the lowest root to one of the indecomposable components of Π and then removing one root from that component. We will say that Π' is obtained from Π by an elementary transformation.

We will first state three results due to Dynkin [3, Theorems 5.1–5.3].

Proposition 1. *Let C be a π-system in a root system R. Then $[C]$ is a root subsystem of R with basis C.*

Proposition 2. *Let C be a π-system in an indecomposable root system R of rank n. Then C can be extended to a π-system, D, with n elements.*

Proposition 3. *Let D be a π-system with n elements in an indecomposable root system R of rank n. Then D can be obtained by a sequence of elementary transformations of a basis of R.*

This shows that extending C will not always give us a basis of R, but that the extension can be obtained by a sequence of elementary transformations of a basis of R.

The next three propositions are due to Bourbaki [2, Chapter 6, Section 1, Corollary to Proposition 4, Corollary 4 to Proposition 20 and Proposition 24].

Proposition 4. *Let V' be a subspace of V and let V'' be the subspace spanned by $R' = R \cap V'$. Then R' is a root system in V''.*

Proposition 5. *If B' a subset of a basis B of R and V' the subspace of V spanned by B', then B' is a basis of the root system $R' = R \cap V'$.*

Proposition 6. *Let B' be a basis of $R' = R \cap V'$, where V' is a subspace of V. Then B' can be extended to a basis B of R and R' is the set of roots in R that are linear combinations of elements of B'.*

Then next result is a simple combination of Propositions 1–6.

Theorem 7. *Let C be a π-system in R and let V' the subspace of V spanned by C. Then $[C]$ is a root subsystem of $R' = R \cap V'$, C can be obtained from a basis of R' by a sequence of elementary transformations, and C can be extended to a basis B of R if and only if C is a basis of R', i.e., $[C] = R'$.*

3. Extension results

It follows from Theorem 7 that C will extend to a basis of R unless there is a root system R' such that

$$[C] \subset R' \subset R,$$

where $\operatorname{rank}[C] = \operatorname{rank} R' < \operatorname{rank} R$, $[C] \neq R'$ and C can be obtained from a basis of R' by a sequence of elementary transformations. We are for simplicity assuming that $[C]$ is indecomposable, and hence R' is also indecomposable. The next lemma is proved by inspection of the extended Dynkin diagrams.

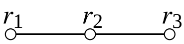
Lemma 8. *The only indecomposable roots systems $[C] \subset R'$ where C can be obtained by a sequence of elementary transformations of a basis B' of R' are listed below*; *$D_4 \subset F_4$ is obtained by using two elementary transformations, while all the other only require one*:

$$A_3 \subset B_3, \qquad D_n \subset B_n \quad \textit{for } n \geqslant 4, \qquad A_7 \subset E_7,$$
$$A_8 \subset E_8, \qquad D_8 \subset E_8, \qquad B_4 \subset F_4, \qquad D_4 \subset F_4, \qquad A_2 \subset G_2.$$

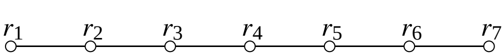
Since our pairs must sit inside a root system, R, of higher rank, and the diagram of C must be a subdiagram of the diagram of R, there are only two cases that satisfies our conditions, $A_3 \subset B_3 \subset B_n$ and $A_7 \subset E_7 \subset E_8$. We must also be able to tell whether C really sits inside the extended diagram of R'. In that case we can get outside of $[C]$ by taking linear combinations of the roots in C, as described in the next theorem.

Theorem 9. *An indecomposable π-system C in R can be extended to a basis B of R, unless $[C] \subset R$ is on of the following two cases*:

1. *$A_3 \subset B_n$ for $n \geqslant 4$ and $C = \{r_1, r_2, r_3\}$ has Dynkin diagram*



   and $(r_1 + 2r_2 + r_3)/2$ is a root in B_n.
2. *$A_7 \subset E_8$ and $C = \{r_1, \ldots, r_7\}$ has Dynkin diagram*



and either

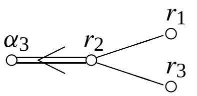
$$(r_7 + r_1 + 2r_2 + 3r_3 + 4r_4 + 3r_5 + 2r_6)/2 \quad \textit{or}$$
$$(r_1 + r_7 + 2r_6 + 3r_5 + 4r_4 + 3r_3 + 2r_1)/2$$

is a root in E_8.

Proof. We cannot use the pairs in Theorem 8 involving E_8, F_4 and G_2, because we cannot fit them into any bigger root systems. We also cannot fit a D_k diagram inside B_n, so we are left with $A_3 \subset B_3 \subset B_n$ and $A_7 \subset E_7 \subset E_8$.

We will use the bases for the root systems listed in [2] and denote the lowest root by α_0. The only way C can fail to extend is if C is obtained from a R' diagram by an elementary transformation, in which case we can get outside of $[C]$ by taking linear combinations of the roots in C.

For $A_3 \subset B_n$, either A_3 is be the Y-branch at the end of the extended diagram of B_3 or A_3 is part of the diagram of B_n. In the first case, $r_2 = \alpha_2$ and either r_1 or r_3 must be the lowest root $-(\alpha_1 + 2\alpha_2 + 2\alpha_3)$ and the other must be α_1. In either case, $(r_1 + 2r_2 + r_3)/2 = -\alpha_3 = -e_3$, which is a root of B_n. So V' is the span of $\{e_1, e_2, e_3\}$, and $R' = B_3$ while $[C] = A_3$. Hence C cannot be extended to a basis of B_n by Theorem 7.



However, if $C = \{\alpha_1, \alpha_2, \alpha_3\}$ corresponds to an A_3 that is a subdiagram of the diagram of B_n, then $(r_1 + 2r_2 + r_3)/2$ is not even a root of A_n. In this case V' does not contain any short roots, so $R' = [C]$ and C can be extended to a basis of B_n.

For $A_7 \subset E_8$, the A_7 is either part of the diagram of E_8 or is part of the extended diagram of E_7. We can assume that C is either of the form

$$C = \{\alpha_1, \ldots, \alpha_8\} - \{e_2\} \quad \text{or} \quad C = \{\alpha_0, \alpha_1, \ldots \alpha_7\} - \{e_2\},$$

where $\alpha_0 = -2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7$ is the lowest root of E_7. In the second case we do not know whether r_1 or r_7 is the extended root, but by an argument similar to above, either

$$(r_7 + r_1 + 2r_2 + 3r_3 + 4r_4 + 3r_5 + 2r_6)/2 \quad \text{or}$$
$$(r_1 + r_7 + 2r_6 + 3r_5 + 4r_4 + 3r_3 + 2r_1)/2$$

will be equal to $-\alpha_2$, while in the first case neither will be a root. The reason why we need to look at both expressions is because we do not know the orientation of the A_7 diagram inside the extended E_7 diagram. It follows that in the first case C can be extended to a basis of E_8, while in the second case C can only be extended to a basis of A_8. $\square$

Notice that $C = \{\alpha_1, \alpha_2, \alpha_0, \} = \{e_1 - e_2, e_2 - e_3, -e_1 - e_2\}$, where α_0 is the lowest short root in C_n is *not* a π-system in C_n, since $e_1 - e_2 - (-e_1 - e_2) = 2e_1$ is a root in C_n. We initially considered linearly independent sets of roots with nonpositive inner products, i.e., linearly independent, admissible sets of roots instead of π-systems. π-systems are always admissible, and for simply laced root systems, indecomposable π-systems are admissible, since the only way the difference between two roots can have the same length as the two roots is if the angle between them is $\pi/3$. However, $C \subset C_n$ shows that this is false for multiply-laced root systems. In particular, [4, Exercise 34, p. 177] appears to be incorrect. (They do not require π-systems to be linearly independent, but that does not make any difference.)

Notice that $D_k \subset B_k$ is listed in Lemma 8, while $D_k \subset C_k$ is not. There are two standard ways of constructing equal rank inclusions of root systems. One is to use elementary transformations and is used by Borel and de Siebenthal [1]. The other is to consider the set of short and long roots in multiply-laced root systems. D_k forms the long roots in B_k and the short roots in C_k, but only the B_k inclusion can be obtained by an elementary transformation.

Notice also that we only talk about root systems, and not about Lie subalgebras. Unless we know something about the Cartan subalgebras, inclusions of root systems and inclusions of Lie algebras will not necessarily correspond. The fact that $D_k \subset C_k$ does not imply that $\mathfrak{so}(2n) \subset \mathfrak{sp}(2n)$.

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